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Scaling properties of the eigenvalue spacing distribution for band random matrices

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Abstract. We show that the spacing distribution for eigenvalues of band random matrices is described by a single parameter b^2/N, where b is the band half-width and N is the size of the matrices. It is also shown that the eigenvalue's density obeys the semicircle law. The found scaling behaviour suggests that the fluctuation properties in the intermediate regime, between Wigner–Dyson and Poisson, are universal.

1. Introduction

Random matrix ensembles are extensively used as models for describing the statistical properties of levels of complex systems such as heavy nuclei and many electron atoms. This idea, put forward by Wigner and Dyson [1], proves to be effective for many physical systems and shows once more that in physics symmetries are the relevant features: the fluctuation properties seem to be relatively insensitive to the details of the interaction. It has recently become clear that classically chaotic dynamics is the underlying condition for the random matrix analogy to apply, even for systems with a few degrees of freedom (see e.g. [2]).

However, in quite general situations, such as in discretized models of solid state physics or in perturbed integrable systems, a band structure in the Hamiltonian is a common occurrence [3]. Band random matrix (BRM) ensembles may therefore prove to be more effective than the standard random matrix ensembles. A general semiclassical argument has been given by Feingold *et al* [4] in support of this hypothesis, and some motivation may be traced back to a paper by Chirikov [5].

The mathematical investigation of BRM ensembles is very difficult, since they are not rotationally invariant. In these cases we must rely mostly on numerical computations, with a few exceptions: apart the obvious limit case of the Gaussian orthogonal ensemble (GOE), the simplest analytically investigated case is provided by the other extreme, namely tridiagonal random matrices. The latter describe disordered linear chains and the exact formula for the eigenvalue distribution was found by Dyson [6], and yet it is very complicated. Another analytically studied model is that of 'bordered matrices', in which the off-diagonal elements take randomly the values ± 1 [7].

Only recently have BRM been given much attention. They were investigated by Seligman *et al* [8] as a model for interpolating between Poissonian and Wigner–Dyson statistics. Along this line, Cheon [9] numerically examined the low moments of the

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level spacing distribution. The small spacing behaviour of the joint distribution of eigenvalues for small matrices has been investigated in [10, 11].

In our previous paper [12] we studied the localization properties of the BRM eigenvectors. The motivation for such analysis was based on the analogy with the quantum dynamics of the well-known model of the kicked rotator, for which the band structure appears in the time-evolution operator [13, 14]. We have shown that, unlike the case of GOE matrices, for which the eigenvectors uniformly distribute on the unit sphere (as a consequence of the rotational invariance), for BRM the eigenvectors display a scaling behaviour. More precisely, our main result is the following: the average localization length divided by the size N of the matrices is a function of b^2/N, where b is the band half-width.

A natural question is whether this scaling behaviour is valid also for the statistical properties of the eigenvalues. This paper provides a positive answer to this question, together with numerical evidence that the level spacing distribution for b and $N \to \infty$ only depends on the same scaling parameter b^2/N.

2. The eigenvalue density

A BRM ensemble is defined as the set of real symmetric $N \times N$ matrices with matrix elements $A_{ij} = 0$ for $|i-j| \geq b$. The parameter $1 \leq b \leq N$ is therefore the number of non-zero elements in the first row. The number of independent non-zero matrix elements is

$$F = \tfrac{1}{2} b(2N - b + 1). \tag{1}$$

They are chosen as independent random variables with Gaussian distributions:

$$\begin{aligned} P(A_{ii}) &= \sqrt{\omega/\pi}\exp(-\omega A_{ii}^2) \\ P(A_{ij}) &= \sqrt{2\omega/\pi}\exp(-2\omega A_{ij}^2) \qquad i < j\,. \end{aligned} \tag{2}$$

The ensemble is fully characterized by the three parameters ω, b and N; however, the first parameter only determines the size of the eigenvalues and is not relevant for describing statistical properties. For $b = N$ the matrix ensemble coincides with GOE, while for $b = 2$ and 1 we have tridiagonal and diagonal random matrices, respectively.

In order to gain some insight in the properties of the eigenvalue density $\rho(\lambda)$ of BRM, we start with the analysis of its low moments, which may be evaluated analytically as ensemble averages:

$$\langle\lambda^{2n}\rangle = \int \lambda^{2n}\rho(\lambda)\,\mathrm{d}\lambda = Z^{-1}\int [\mathrm{d}A]\frac{1}{N}\mathrm{Tr}(A^{2n})\exp(-\omega\mathrm{Tr}A^2) \tag{3}$$

where

$$[\mathrm{d}A] = \prod_{i-j<b}\mathrm{d}A_{ij} \qquad Z = \int [\mathrm{d}A]\exp(-\omega\mathrm{Tr}A^2). \tag{4}$$

The odd moments are evidently zero. The case $n = 1$ is not of particular interest, since it merely amounts to a counting of the non-zero matrix elements

$$\langle\lambda^2\rangle = -\frac{1}{N}\frac{\partial}{\partial\omega}\log Z = \frac{1}{2N\omega}F. \tag{5}$$

The higher moments involve some combinatorial work and explicitly take into account the band structure. The result for $n = 2$ is

$$\langle\lambda^4\rangle = \frac{1}{8N\omega^2}(11F + G - 5N) \tag{6}$$

where

$$G = \begin{cases} 2N(b-1)(2b-3) - \frac{2}{3}b(b-1)(5b-7) & 2b \leq N \\ N(N-1)(N-2) - \frac{2}{3}(N-b)(N-b+1)(2N+b-5) & 2b > N. \end{cases} \tag{7}$$

The twofold behaviour arises because of corner and finite-size effects. To avoid them, we shall restrict ourselves, in the following, to the case $b < N/2$.

It is interesting to investigate the adimensional (i.e. ω-independent) ratio

$$\eta(b, N) = \frac{\langle\lambda^4\rangle - 2\langle\lambda^2\rangle^2}{\langle\lambda^2\rangle^2}. \tag{8}$$

This ratio can be shown to be identically zero for a semicircle distribution of the eigenvalues and has value one for a Gaussian distribution, which is the case $b = 1$. For the GOE ensemble it is equal to $(N+3)/(N+1)^2$ and therefore it vanishes for increasing N, consistently with the semicircular limit distribution for GOE.

We now consider the ratio (8) for BRM in the limit of large N and b. Taking the limit such that $b/N \to 0$, the ratio η goes to zero with the asymptotics

$$\eta \approx \left(\frac{1}{2b} + \frac{b}{3N}\right). \tag{9}$$

The next adimensional ratio $\langle\lambda^6\rangle/\langle\lambda^2\rangle^3$ has a much more complicated dependence on the band structure. Nevertheless it can be shown that for both N and b going to infinity, $b/N \to 0$, the ratio approaches the value 5, which corresponds to the semicircular distribution.

The above discussion leads us to conjecture a semicircle distribution for BRM ensembles for large N and b. The normalized semicircle distribution can be written in the form

$$\rho(\lambda) = \frac{2}{\pi r^2}\sqrt{r^2 - \lambda^2}. \tag{10}$$

It has second moment $\langle\lambda^2\rangle = r^2/4$; therefore, from expression (5) one obtains $r^2 = 2F/(N\omega)$.

For finite but large N we have numerically found that the eigenvalue's distribution is very close to the semicircle law (10). An example is given in figure 1, where the distribution $\rho(\lambda)$ is computed from the eigenvalues of six matrices of size $N = 3200$ and $b = 69$.

A more accurate comparison can be made by computing the moments of the numerically found eigenvalue distribution. In table 1 we give the values of the adimensional ratios $\langle\lambda^{2n}\rangle/\langle\lambda^2\rangle^n$ for different N at the fixed value $b^2/N = 3/2$, together with the corresponding values which result from the semicircle law.

As is seen from the table, for large N, these two ratios are quite close. Notice that the convergence to the semicircle values becomes worse as n increases, thus indicating that, for finite N, higher moments are more sensitive to the edges of the distribution.

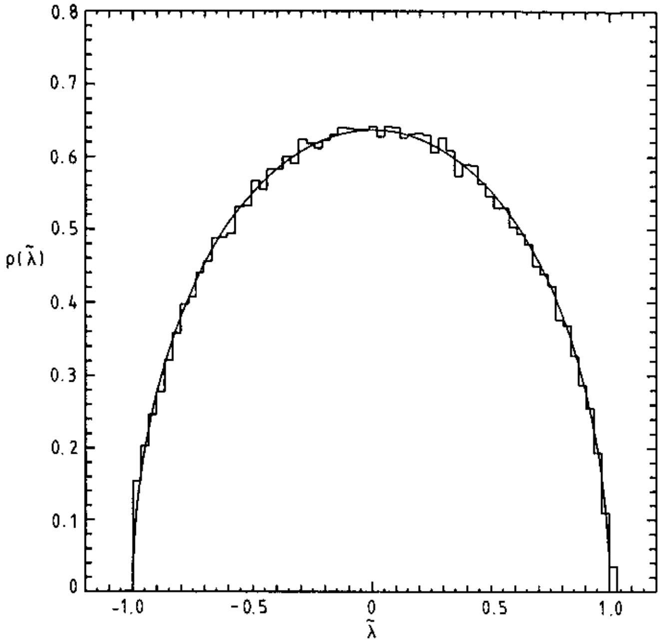

Figure 1. Histogram of the computed density of states $\rho(\tilde{\lambda})$ as a function of the rescaled parameter $\tilde{\lambda} = \lambda/r$. Here six matrices with $N = 3200$, $b = 69$, $\omega = 1$ have been used. The curve for the semicircle law (10) is also shown.

Table 1.

n	$N = 400$	$N = 800$	$N = 1600$	$N = 3200$	Semicircle
2	2.042	2.029	2.022	2.016	2
3	5.276	5.198	5.144	5.100	5
4	15.42	15.02	14.72	14.50	14
5	48.67	46.75	45.27	44.28	42
6	162.1	153.2	146.2	142.0	132

3. The spacing distribution

As mentioned in the introduction, the investigation of the structure of the eigenfunctions of BRM has led to the discovery of the scaling parameter $x = b^2/N$ [12]. More precisely, by an appropriate definition of localization length, the so-called 'entropy localization length' l_H, it has been found numerically that the ratio l_H/N is a function of the scaling parameter x only. On intuitive grounds, one may expect the existence of the scaling property due to the random structure of the eigenstates. Indeed, numerical data show that in the case of strong localization ($1 \ll l_H \ll N$), the eigenstates are random on the scale of their localization length [12, 15]. It was also found that for $x \gg 1$ all eigenstates may be regarded as completely random, as in the case of full random matrices, even if $b \ll N$.

The same scaling properties of eigenfunctions have been earlier found in a model with no random parameters: the kicked rotator on the torus [13–16]. In spite of the strong difference in global properties like the density of states, it was established that this dynamical model and BRM have strong similarities in the statistical properties of spectra and in the structure of eigenfunctions. This similarity is related to the band structure of the band structure of the unitary time evolution operator of the quantum kicked rotator and to the fact that some sort of pseudorandomness appears in the matrix elements of this operator, due to strong chaotic properties of the corresponding classical motion.

Extensive study of this dynamical model [14, 17] has shown that the scaling properties of eigenfunctions are strongly related to the universal fluctuation properties in the quasienergy spectrum. This conjecture has been confirmed in recent numerical experiments [14, 15] on the kicked rotator model. As a consequence, it is natural to expect that the distribution $P(s)$ of spacings between neighbouring eigenvalues of BRM is essentially dependent on the parameter x only, rather than on b and N independently. As is known, in the extreme case of diagonal matrices ($b = 1$) the spacings between eigenvalues are not correlated, resulting in the Poisson distribution for $P(s)$. On the other hand, in the opposite case of fully random matrices ($b = N$), the RMT predicts a specific form of $P(s)$ which is approximately described by the well-known Wigner surmise [15]

$$P(s) = \tfrac{1}{2}\pi s \exp(-\tfrac{1}{4}\pi s^2). \tag{11}$$

An important question is how to describe the intermediate situation for BRM where $P(s)$ changes from the Poisson to the Wigner–Dyson distribution. Taking advantage of the analogy with the kicked rotator model, we follow the approach developed in [14, 15] and assume that the distribution $P(s)$ may be described by the phenomenological formula [18]

$$P(s) = As^{\beta}(1 + B\beta s)^{f(\beta)} \exp\left[-\frac{\pi^2}{16}\beta s^2 - \frac{\pi}{2}\left(1 - \frac{\beta}{2}\right)s\right] \tag{12}$$

where A and B are normalizing parameters and

$$f(\beta) = \frac{2^{\beta}(1 - \frac{1}{2}\beta)}{\beta} - 0.16874\,.$$

For $\beta = 0$ the expression (12) reduces to the Poisson distribution. For $\beta = 1,\ 2,\ 4$ it approximates very closely the $P(s)$ distribution for Gaussian orthogonal, unitary and symplectic ensembles (GOE, GUE, GSE). Expression (12) is more complicated than the one used in [12, 14], but it gives a much better correspondence with RMT predictions. For example, for $\beta = 1$ the deviation from the exact dependence of $P(s)$ (see [19]) is less than 0.3 % for small ($s \leq 0.1$) and large ($s \geq 2$) spacings; it is less than 0.02 % in the most important intermediate region $0.5 \leq s \leq 1.6$. This distribution is thus closer to the exact one than Wigner's distribution (11) itself. The agreement with RMT is very good also for $\beta = 2, 4$. In addition, the dependence (12) seems to be more suitable to fit the numerical data for the intermediate statistics $P(s)$ than the commonly used Brody distribution [20]. Indeed, the latter dependence has definitely a wrong limit for $\beta = 1$ and large spacings $s \gg 1$. Moreover, when using the Brody distribution to fit GOE, one obtains the wrong value $\beta = 0.95$, instead of $\beta = 1$. In addition, Brody's distribution is not valid for situations where the repulsion is larger than 1 (for example, for GUE and GSE).

In our numerical experiments we used BRM with sizes $N = 400$, 800, 1600 and different band sizes $b \gg 1$. The distribution $P(s)$ is obtained by averaging over the $P(s)$ for Q different random matrices with the same N and b ($Q = 50$, 25, 12 for $N = 400$, 800, 1600, respectively). Since the eigenvalue density is not uniform, the spacings have been normalized to the local density. To avoid the influence of large fluctuations caused by the finite size of matrices, a number of eigenvalues at the edges of the semicircle distribution (3) have not been taken into account. As a result, for each N and b, the total number of spacings in the final distribution of $P(s)$ is approximately equal to $M = 16\,000$–$17\,000$.

A few examples of $P(s)$ with the best fit (full curve) of the proposed dependence (12) are presented in figure 2. Here, the parameter x is taken to be approximately constant, $x \approx 1.0$, while the band size b and the size N of the matrices vary. The data give good evidence for the scaling behaviour of the spacing distribution $P(s)$. To show the accuracy of the fit, two curves are also drawn, corresponding to the the 1%-confidence level.

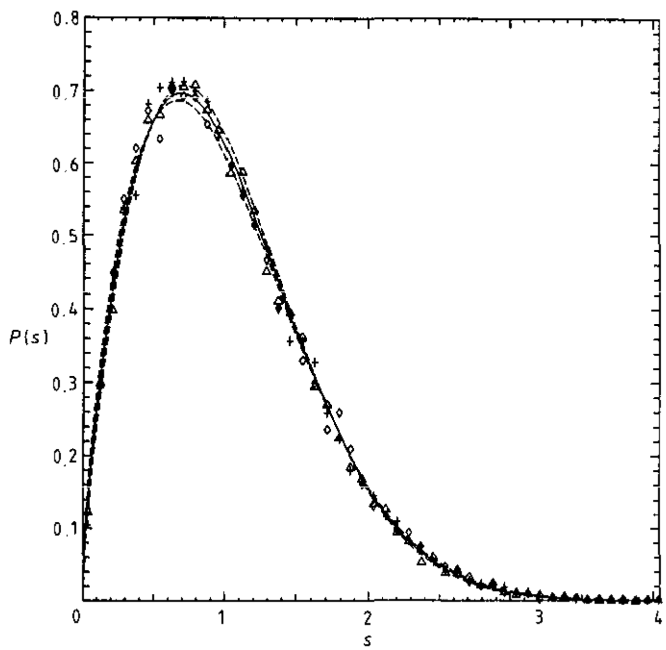

Figure 2. The level spacing distribution $P(s)$ for $x = b^2/N \approx 1$ with $N = 400$ (+), $N = 800$ ($\triangle$), $N = 1600$ ($\diamond$). The full curve corresponds to the expression (12) with the best fitting value $\beta = 0.703$ found for $N = 800$. The broken curves give the lower and upper bounds for 1% confidence level with $\beta_- = 0.620$ and $\beta_+ = 0.759$, respectively.

The summarized data for different values of x are given in figure 3. It is seen that the scaling behaviour for the repulsion parameter β occurs in a large range of the parameter x. This result indicates that fluctuations in the eigenvalue spectra of BRM appear to have universal properties which can be described by a single parameter x.

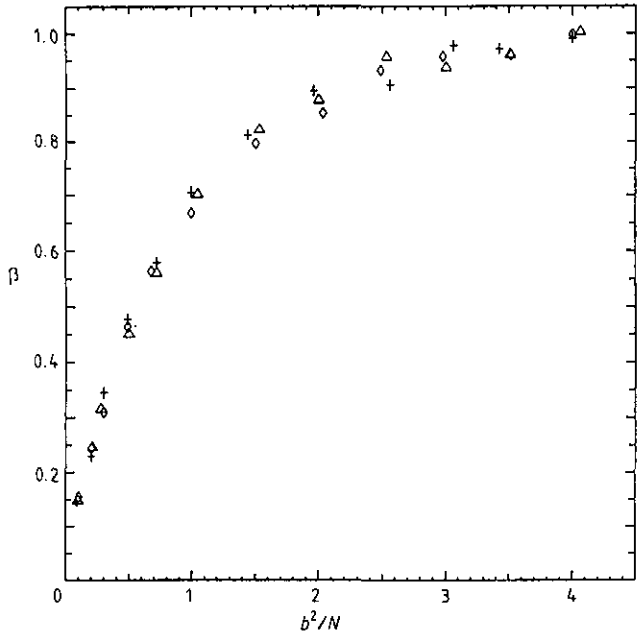

Figure 3. The repulsion parameter β for intermediate statistics $P(s)$ plotted against $x = b^2/N$, for different values of N and b. $N = 400$ (+), $N = 800$ (Δ), $N = 1600$ ($\diamond$). Each value of β was obtained by fitting the numerical data for level spacing distribution with the expression (12). All values of β are within a 1% confidence level.

4. Conclusions

In this paper we have studied the statistical properties of eigenvalues of BRM in the limit of large b and N. The numerical analysis leads to two main conclusions:

(a) The density of the eigenvalues obeys the semicircle law. The conditions under which this result can be rigorously proven is currently under investigation [21,22].

(b) The eigenvalue spacing distribution P(s) depends only on the scaling parameter $x = b^2/N$. A similar scaling behaviour is displayed by the localization length of eigenvectors [12]. It would be very important to find analytical support for this scaling property.

Several other interesting questions arise. One would like for example to have an analytical derivation for the distribution $P(s)$. Indeed the expression we used (equation (12)) to describe $P(s)$ is empirical and has no rigorous support. It would also be very interesting to relate the repulsion parameter β with the normalized entropy localization length $\beta_H = l_H/N$ which exhibits a similar scaling behaviour with the same parameter x [12].

Scaling properties similar to those described here and in previous papers [12, 13] should be expected in the more realistic situations where the sharp band structure is replaced by a sufficiently fast decay of matrix elements away from the diagonal. A support to the above conclusion can be found in [8, 9].

To conclude this paper, we would like to make an important remark. According to Wigner–Dyson, the fluctuations of spectra of full random matrices have universal properties in the sense that they are shared by different complex quantum systems in spite of the fact that they have, for example, different density of states. On the other hand, full random matrices describe limit situations. Indeed, as mentioned in the introduction, most physical systems are described by matrices with a band structure which reflects the finite range of the interaction. The results presented in this paper lead us to conjecture that in the intermediate case, corresponding to a level spacing distribution between Poissonian and Wigner–Dyson, fluctuation properties have a universal character. For example, as we have shown, the kicked rotator model on the torus and BRM have similar fluctuation properties both for spectra and eigenfunctions, in spite of the fact that the densities of states are completely different (semicircle for BRM and uniform distribution of quasi-energies of KR).

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